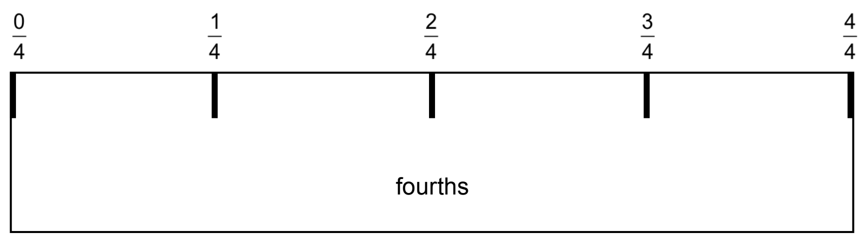
MODELS FOR FRACTIONS

Linear Models

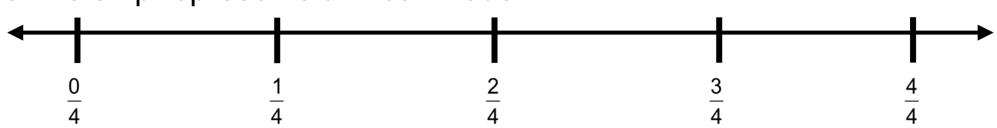
One useful model for fractions is the liner model. In a linear model, the whole (or unit) is represented by a specified interval on a number line. Then fractions are represented as lengths of intervals in comparison to the length of the whole.

The paper strip pictured below represents 1 whole unit of length, divided into fourths (four equal units of length). Notice that the very left edge represents zero, and the very right edge represents 1. Rulers work in much the same way.

This strip is marked off in fourths.



This edge of the strip represents a linear model.

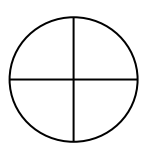


One common error in working with linear models is to start counting “1” at the very left edge, or to count tick marks instead of “spaces.” Notice that it requires 5 tick marks to make 4 spaces.

Area Models

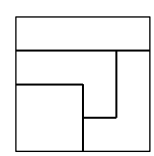
Another useful model for fractions is the area model. In an area model, the whole is represented as the area of some specified shape. Then fractions are represented as areas of shapes that can be compared to the whole.

If the circle is defined as 1 whole, and each part is of equal area, then each part represents $\frac{1}{4}$ of the whole.



These parts happen to be congruent as well.

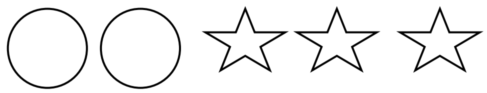
If the rectangle is defined as 1 whole, and each part is of equal area, then each part represents $\frac{1}{4}$ of the whole.



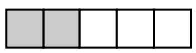
These parts are not all congruent, but they still have equal area.

Set Models

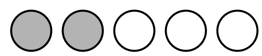
A third useful model for fractions is the set model. Set models are based on numbers of objects in a set, not their area. For example, in this diagram, $\frac{2}{3}$ of the objects are circles, and $\frac{3}{5}$ of the objects are stars.



Sometimes the set model resembles an area model. For example, in the diagram on the left below, $\frac{2}{5}$ of the area of the rectangle is shaded. In the diagram, on the right below, $\frac{2}{5}$ of the circles are shaded.



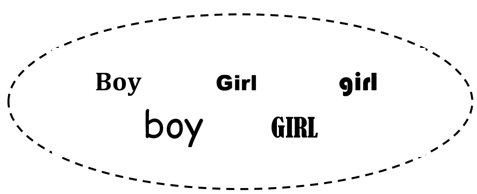
Area model for $\frac{2}{5}$



Set model for $\frac{2}{5}$

In this example, each of the 5 small squares has equal area, and each of the 5 small circles has equal area too. However, in the set model, the fraction is based on the number of shaded circles, not the size of them.

Consider the following set model situation. In a classroom table group, $\frac{2}{5}$ of the students are boys. Does this mean that all of the students have the same area (or volume, or are somehow of equal size)? Of course not. Their common feature is that they are all people.

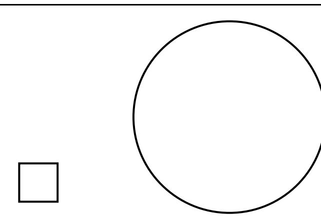


FRACTION ORDERING AND EQUIVALENCE

Sense-Making Strategies for Comparing and Ordering Fractions

Examples	Name	Ordering Strategy
$\frac{1}{3} < \frac{1}{2} < \frac{3}{4}$	Benchmark fractions	Benchmark fractions are fractions that are easily recognizable, such as $\frac{1}{2}$. For example, $\frac{3}{8} < \frac{1}{2}$, because 3 is less than half of 8.
$\frac{1}{8} < \frac{1}{5} < \frac{1}{4}$	Unit fractions	When comparing unit fractions, the fraction with the greater denominator has a smaller value. Think: “When you are very hungry, do you want to share a pizza equally among 8 friends or 4 friends? In which situation do you get more pizza?”
$\frac{3}{8} < \frac{3}{5} < \frac{3}{4}$	Fractions with common numerators	When comparing fractions with common numerators, the fraction with the greater denominator has a smaller value. Using similar reasoning as above: “If ONE-fourth is greater than ONE-eighth, then THREE-fourths must be greater than THREE-eighths.”
$\frac{1}{12} < \frac{3}{12} < \frac{8}{12}$	Fractions with common denominators	When comparing fractions with common denominators, the fraction with the greater numerator has a greater value. Think: “A pizza is divided into 8 equal parts. If you eat 1 slice and your friend eats 3 slices, who ate more pizza?”
$\frac{3}{4} < \frac{4}{5} < \frac{7}{8}$	1 minus a unit fraction	All of these are less than 1 whole by a unit fraction (Think of it as the “missing piece.”) $\frac{7}{8}$ has a smaller piece missing ($\frac{1}{8}$); $\frac{3}{4}$ has a larger piece missing ($\frac{1}{4}$); therefore, $\frac{7}{8} > \frac{3}{4}$.

In these comparisons, we assume that all fractions in each example refer to the same whole. This is important because $\frac{1}{2}$ of the circle to the right has a greater area than $\frac{9}{10}$ of the square to the right.



The Big One

The “big 1” is a notation for 1 in the form of a fraction $\frac{n}{n}$ $(n \neq 0)$. For example,

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \ldots$$

We can use the following picture to help remind us that these fractions are equivalent to 1:

$$1 = \frac{8}{8}$$

The “big 1” can be used to show equivalence of fractions. For example,

$$\frac{2}{5} \times \frac{10}{10} = \frac{20}{50} \quad \text{or} \quad \frac{20}{50} \div \frac{10}{10} = \frac{2}{5}.$$

Why Can’t You Divide by Zero?

<u>Strategy 1</u>

Consider the fact $6 \div 2 = 3$ or $\begin{array}{r} 3 \\ 2\overline{)6} \end{array}$. We can convince ourselves that this is correct, because we know that $2 \bullet 3 = 6$.

Now consider $6 \div 0 = ?$ or $\begin{array}{r} ? \\ 0\overline{)6} \end{array}$. What can be multiplied by 0 to get a result of 6? Nothing!

<u>Strategy 2</u>

Division can be thought of as repeated subtraction.

Consider the same fact $6 \div 2 = 3$ or $\begin{array}{r} 3 \\ 2\overline{)6} \end{array}$.

Rewrite the division statement as follows:

$$\begin{array}{r} 2\overline{)\;\;6} \\ \underline{-2} \\ 4 \\ \underline{-2} \\ 2 \\ \underline{-2} \\ 0 \end{array}$$

We count that there are 3 subtractions of 2 from 6, and then there is nothing remaining to subtract. Done!

Now consider $6 \div 0 = ?$ or $\begin{array}{r} ? \\ 0\overline{)6} \end{array}$.

Rewrite the division statement as follows:

$$\begin{array}{r} 0\overline{)\;\;6} \\ \underline{-0} \\ 6 \\ \underline{-0} \\ 6 \\ \underline{-0} \\ 6 \end{array}$$

etc.

We see that this process will never end!

We conclude that division by zero cannot be performed, and we say that it is <u>undefined</u>.

“Splitting Diagrams” and Equivalent Fractions

A “splitting a diagram” illustrates equivalent fractions. For example, to show that $\frac{1}{2} = \frac{4}{8}$, we split the first diagram into eight equal parts to get the second diagram.

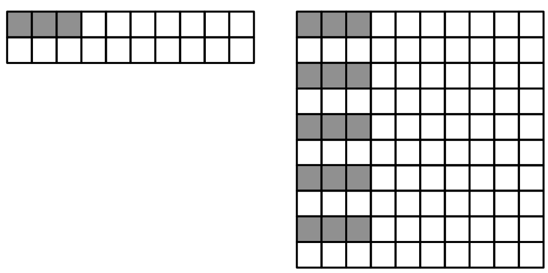
Using the “big 1,” this equivalence can be written:

$$\frac{1}{2} \bullet \frac{4}{4} = \frac{4}{8}.$$

In a “splitting a diagram,” the size of the whole does not change.

“Replicating Diagrams” and Equivalent Fractions

“Replicating patterns” visually illustrate equivalent fractions that have the same fractional amount shaded. For example, to show that $\frac{3}{20} = \frac{15}{100}$ we replicate this 20-square pattern to obtain a 100-square grid.



$$\frac{3}{20} \qquad = \qquad \frac{15}{100}$$

Using the “big 1,” this equivalence can be written:

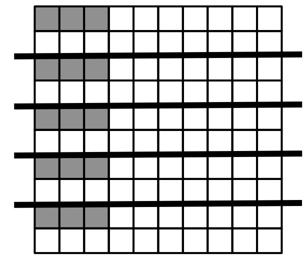
$$\frac{3}{20} \bullet \frac{5}{5} = \frac{15}{100}.$$

Visually, multiplying the numerator by 5 represents replicating the shaded parts five times, and multiplying the denominator by 5 represents replicating the number of parts in the denominator five times.

In a “replicating diagram,” the size of the part does not change.

"Grouping Diagrams" and Equivalent Fractions

This "grouping diagram" illustrates the "undoing" of a replicating diagram:

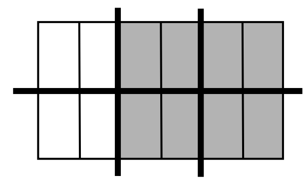


$$\frac{15}{100} = \frac{3}{20}$$

Using the "big 1," this equivalence can be written:

$$\frac{15}{100} \div \frac{5}{5} = \frac{3}{20}$$

This grouping diagram illustrates the "undoing" of a splitting diagram:



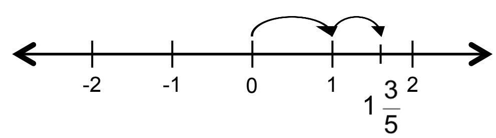
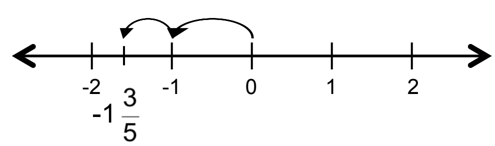
$$\frac{8}{12} = \frac{4}{6}$$

Using the "big 1," this equivalence can be written:

$$\frac{8}{12} \div \frac{2}{2} = \frac{4}{6}$$

Mixed Numbers and the Number Line

Breaking numbers into parts sometimes makes them easier to manipulate. For example, thinking about 57 as a combination of 50 and 7 might make it easier to add it to other numbers. This can be helpful with mixed numbers and their opposites as well.

Traditional notation "shorthand"	Expanded notation "longhand"	Number line representation
$1\frac{3}{5}$	$1 + \frac{3}{5}$	
$-1\frac{3}{5}$	$-(1 + \frac{3}{5}) = -1 - \frac{3}{5}$	

Error Alert: Do not rewrite $-1\frac{3}{5}$ as $-1 + \frac{3}{5}$. This has a different value.

Improper Fractions and Mixed Numbers

Improper fractions may be represented as mixed numbers and vice versa.

Example: Change $5\frac{7}{8}$ into an improper fraction.

Since $5 = \frac{40}{8}$, $5\frac{7}{8} = \frac{40}{8} + \frac{7}{8} = \frac{47}{8}$.

Here is a shortcut.

Think: "5 times 8 is 40, and 40 + 7 is 47.

So $5\frac{7}{8} = \frac{47}{8}$."

$5 \times 8 = 40$ $\quad 5\frac{7}{8} \quad$ $40 + 7 = 47$

(40 *eighths*) (47 *eighths*)

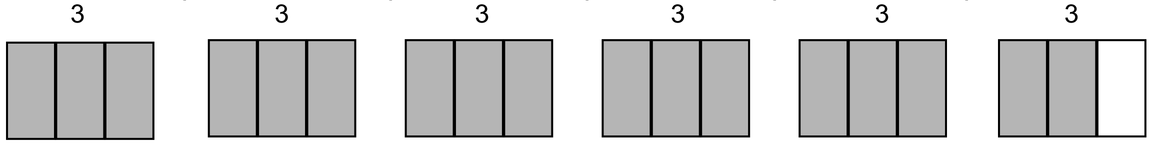
Example: Change $\frac{17}{3}$ into a mixed number.

Recall that $\frac{17}{3}$ can be written as $17 \div 3$

$17 \div 3 = 5$, with a remainder of 2

$$\frac{17}{3} = \frac{15}{3} + \frac{2}{3} = 5 + \frac{2}{3} = 5\frac{2}{3}$$

$$\frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{2}{3} = \frac{17}{3}$$



$$1 + 1 + 1 + 1 + 1 + \frac{2}{3} = 5\frac{2}{3}$$